Midterm Exam – Time: 90 Minutes

Problem 1 (15 points)

You roll a die 5 times. What is the probability that at least one value is observed more than once?

Solution:

It is easier to solve this using the complement concept. Let A be the event that at least one value is observed more than once. So, P(A) is what we need to find. The complement of A, then is the event in which no repetition is observed. If you roll a die 5 time you will have 6.6.6.6.6 = 6^5 possible outcomes. For the first roll, any one of the 6 outcomes in accepted, but for the second roll only 5 is accepted because we do not want to repeat what we had in the first roll. For the third roll, only 4 possible outcomes are accepted and so on because we do not want to repeat the values obtained in previous rolls. So, we have 6.5.4.3.2.1 possibilities. Therefore,

$$\boxed{P(A^c) = \frac{6*5*4*3*2*1}{6*6*6*6*6*6} = \frac{5}{54}}$$

Then P(A) is simply, 1 - $P(A^c)$ = 1 – 5/54 = 49/54

Problem 2 (10 points)

What should the values of c and d be for the following function to be a legitimate probability distribution function if we know that the E(X) = 5/2?

$$P_X(k) = \begin{cases} \mathbf{c} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \mathbf{d} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

We use the two concepts that we have learned. First, for $P_X(k)$ to be a legitimate probability distribution, it has to abide by the probability axioms. Sum of all probabilities have to equal to 1.

$$\boxed{c + \frac{1}{8} + \frac{1}{3} + d = c + d + \frac{11}{24} = 1} \quad \text{or,} \quad c + d = \frac{13}{24}$$

E(X) is given as 5/2. So we calculate the E(X) based on its definition and set it equal to the given value.

$$E(x) = \sum_i x_i P_i = 1 * c + 2 * \left(\frac{1}{8}\right) + 3 * \left(\frac{1}{3}\right) + 4 * d = \frac{5}{2}$$

Or,

$$c + 4d + \frac{5}{4} = \frac{5}{2} \Rightarrow c + 4d = \frac{5}{4}$$

Solving the linear system of equation for c and d we get c = 22/72 and d = 17/72.

Since both values are between 0 and 1, this is a legitimate distribution function.

Problem 3 (20 Points)

Let X be a discrete random variable with the following PMF

$$P_X(k) = \begin{cases} 0.5 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.2 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find EX.

(b) Find $\text{Var}(X)$, and $SD(X)$.

(c) If $Y = \frac{2}{X}$, find EY.

Solution:

$$\begin{aligned}EX &= \sum_k x_k P_x(x_k) \\ &= 1 \times 0.5 + 2 \times 0.3 + 3 \times 0.2 = 0.5 + 0.6 + 0.6 = 1.7\end{aligned}$$

$$\begin{aligned}EX^2 &= \sum_k x_k^2 P_x(x_k) \\ &= (1)^2 \times 0.5 + (2)^2 \times 0.3 + (3)^2 \times 0.2 = 0.5 + 1.2 + 1.8 = 3.5\end{aligned}$$

Thus,

$$var(X) = EX^2 - (EX)^2 = 3.5 - (1.7)^2 = 0.61$$

$$SD(X) = \sqrt{var(X)} = \sqrt{0.61} = 0.781$$

$$\begin{aligned}E[\frac{2}{X}] &= \sum_k (\frac{2}{x_k}) P_x(x_k) \\ &= (\frac{2}{1}) \times 0.5 + (\frac{2}{2}) \times 0.3 + (\frac{2}{3}) \times 0.2 \\ &= 1 + 0.3 + \frac{0.4}{3} = 1 + \frac{3}{10} + \frac{2}{15} = \frac{43}{30}\end{aligned}$$

Problem 4 (20 Points)

Let $X \sim N(3, 9)$.

a. Find $P(X > 0)$.
b. Find $P(-3 < X < 8)$.
c. Find $P(X > 5|X > 3)$.

Solution:

To find P(X>0), we will find 1 – P (X ≤ 0). We know that $\mu = 3$, $\sigma^2 = 9$, so $\sigma = 3$ and therefore, $Z = (x - \mu)/\sigma = (0 - 3)/3 = -1$. Standard Normal table is provided for positive values of Z. We can use symmetry as demonstrated in the class and find the value for P (Z≤ 1). The find its complement, 1- P (Z≤ 1) which would give us P (Z ≥1) which is the same as P (Z≤ -1).

	0.00	0.01	0.02	
0.8	0.7881	0.7910	0.7939	0.
0.9	0.8159	0.8186	0.8212	0.
1.0	$\boxed{0.8413}$	0.8438	0.8461	0.
1.1	0.8643	0.8665	0.8686	0.

P(X>0) = 1 – P (X ≤ 0) = 1 – [1 – P (Z ≤ 1)] = P (Z ≤ 1) = 0.8413

P (−3 < X < 8) = F_X (8) − F_X (-3) = F_Z [(8−3)/3] - F_Z [(−3-3)/3] = F_Z (5/3) - F_Z (-2) = F_Z (5/3) – [1 - F_Z (2)] = F_Z (5/3) -1 + F_Z (2)

5/3 can be directly calculated from the table and for -2, we use the method shown above. We find the value for P (Z ≤ 2)

For Z=2, table value is 0.9772, for 5/3 = 2.6666 we need to interpolate.

For 2.66 we have 0.9961 and for 2.67 we have 0.9962 which interpolation will lead to smaller decimals beyond the 4 digit so we can accept either values. Here I chose 0.9962.

P (−3 < X < 8) = 0.9962 – 1 + 0.9772 = 0.9734

$$
\begin{aligned}
P(X > 5 | X > 3) &= \frac{P(X > 5, X > 3)}{P(X > 3)} \\
&= \frac{P(X > 5)}{P(X > 3)}
\end{aligned}
$$

= [1-P(X≤5)] / [1-P(X≤3)] = [1- F_Z [(5−3)/3] / [1- F_Z [(3−3)/3] = [1- F_Z (2/3)] / [1- F_Z (0)]

Similar steps as before to find the final value.

Problem 5 (15 Points)

X ~ Poi (4.8). Find P (3 ≤ X ≤ 7).

$X \sim Poisson(\lambda)$, if its range is $R_X = \{0, 1, 2, 3, \ldots\}$, and its PMF is given by

$$P_X(k) = \begin{cases} \frac{e^{-\lambda}\lambda^k}{k!} & \text{for } k \in R_X \\ 0 & \text{otherwise} \end{cases}$$

Since the distribution is discrete,

P (3 ≤ X ≤ 7) = P(X=4) + P(X=4) + P(X=5) + P(X=6) + P(X=7) =

$$\frac{e^{-4.8}(4.8)^3}{3!} + \frac{e^{-4.8}(4.8)^4}{4!} + \frac{e^{-4.8}(4.8)^5}{5!} + \frac{e^{-4.8}(4.8)^6}{6!} + \frac{e^{-4.8}(4.8)^7}{7!}$$

= 0.1517 + 0.1820 + 0.1747 + 0.1398 + 0.0958 = 0.7441 (rounded to 4 decimals)

Problem 6 (20points)

Let $X \sim Exponential(4)$ and $Y = 4 + 3X$.

a. Find $P(X > 2)$.
b. Find EY and $Var(Y)$.
c. Find $P(X > 2 | Y < 11)$.

We have

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$ and $$F_X(x) = 1 - e^{-\lambda x}$$

So,

$$f_X(x) = 4e^{-4x} \; and \; F_X(x) = 1 - e^{-4x}$$

Thus,

P (X > 2) = 1 – P (X ≤ 2) = 1 – $F_x(2)$ = 1 - (1 – e^{-8})= 1 - 0.9996 = 0.0004

We know that,

If $X \sim Exponential(\lambda)$, then $EX = \frac{1}{\lambda}$ and $\text{Var}(X) = \frac{1}{\lambda^2}$.

Y = 4 + 3X → E(Y) = E (4 + 3X) = 4 + 3E(X) = 4 + 3/4 = 19/4

V (Y) = V (4 + 3X) = V (4) + V (3X) = 0 + 9 V (X) = 9/16

P (X > 2 | Y < 11) = P (X > 2 | 4+3X < 11) = P (X > 2 | X < 7/3)

Remember,

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

P (X > 2 | X < 7/3) = P (X > 2 and X < 7/3) / P (X < 7/3)

P (X < 7/3) = P (X ≤ 7/3) (Continuous distribution) = $1 - e^{-4(7/3)}$ = 0.9999

P (X > 2 and X < 7/3) = P (2 < X < 7/3) = P (2 ≤ X ≤ 7/3) {Continuous distribution) =

$F_x(7/3) - F_x(2) = 1 - e^{-4(7/3)} - [1 - e^{-4(2)}]$ = - 0.9996 + 0.9999 = 0.0003