Assignment 4: Likelihood Function/Maximum Likelihood Estimator

Resource:

H. Pishro-Nik, "Introduction to probability, statistics, and random processes", available at https://www.probabilitycourse.com, Kappa Research LLC, 2014.

Helpful Reminder: Often when working with maximum likelihood functions, out of ease, we maximize the log-likelihood rather than the likelihood to find the maximum likelihood estimator.

Question 1: (20 points)

Let $X_1, \ldots, X_4$ be a random sample from an Exponential(θ) distribution. Suppose we observed $(x_1, x_2, x_3, x_4) = (2.35, 1.55, 3.25, 2.65)$. Find the likelihood function. What is the best estimate for θ?

Solution: If $X_i \sim Exponential(\theta)$, then

$$f_{X_i}(x;\theta) = \theta e^{-\theta x}$$

Thus, for $x_i \geq 0$, we can write

$$\begin{aligned} L(x_1, x_2, x_3, x_4; \theta) &= f_{X_1X_2X_3X_4}(x_1, x_2, x_3, x_4; \theta) \\ &= f_{X_1}(x_1;\theta) f_{X_2}(x_2;\theta) f_{X_3}(x_3;\theta) f_{X_4}(x_4;\theta) \\ &= \theta^4 e^{-(x_1+x_2+x_3+x_4)\theta}. \end{aligned}$$

Since we have observed $(x_1, x_2, x_3, x_4) = (2.35, 1.55, 3.25, 2.65)$, we have

$$L(2.35, 1.55, 3.25, 2.65; \theta) = \theta^4 e^{-9.8\theta}.$$

To find the best estimate for θ we need to differentiate the above function and set it equal to zero.

$$\frac{dL}{d\theta} = \frac{d(\theta^4 e^{-9.8\theta})}{d\theta} = 4\theta^3 e^{-9.8\theta} - 9.8 e^{-9.8\theta} = e^{-9.8\theta}(4\theta^3 - 9.8) = 0$$

$$e^{-9.8\theta} > 0 \ so\ the\ only\ answer\ comes\ from\ (4\theta^3 - 9.8) = 0$$

$$(4\theta^3 - 9.8) = 0 \ \rightarrow 4\theta^3 = 9.8 \ \rightarrow \dot{\theta} = \sqrt[3]{\frac{9.8}{4}} = 1.348$$

Question 2: (20 points)

Let $X_1, \ldots, X_4$ be a random sample from a Geometric(θ) distribution. Suppose we observed $(x_1, x_2, x_3, x_4) = (2, 3, 3, 5)$. Find the likelihood function. What is the best estimate for θ?

Solution:

Since $P(x) = p.(1 - p)^{x-1}$, then

The likelihood function is

$$\begin{aligned} L(x_1, \ldots, x_4; p) &= P_{X_1,\ldots,X_4}(x_1, \ldots, x_4; p) \\ &= P_{X_1}(x_1; p) \cdots P_{X_4}(x_4; p) \\ &= p^4(1-p)^{\sum_{i=1}^{4} x_i - 4} \end{aligned}$$

Now if we plug in our data,

$$p^4(1-p)^{(2+3+3+5)-4} = p^4(1-p)^9.$$

$$\frac{dL}{dp} = \frac{d[\mathrm{p}^4(1-p)^9]}{dp} = 4\mathrm{p}^3(1-p)^9 - 9\mathrm{p}^4(1-p)^8 =$$

$$\mathrm{p}^3(1-p)^8[4(1-p) - 9p] = 0$$

$P^3 = 0 \rightarrow \hat{p} = 0$

$(1 - P)^8 = 0 \rightarrow \hat{p} = 1$

$4(1 - P) - 9p = 0 \rightarrow 4 - 13p = 0 \rightarrow \hat{p} = 4/13 \approx 0.30$

The first two answers are not acceptable so the best estimate is 0.30

Question 3: (30 points)

Let $X_1, \ldots, X_5$ be a random sample from a Poisson(θ) distribution. Suppose we observed $(x_1, x_2, x_3, x_4, x_5) = (2, 3, 1, 4, 3)$. Find the likelihood function. What is the best estimate for θ?

Solution:

The likelihood function is

$$\begin{aligned} L(x_1, \ldots, x_n; \lambda) &= \prod_{i=1}^{n} P_{X_i}(x_1, \ldots, x_n; \lambda) \\ &= \prod_{i=1}^{n} \frac{e^{-\lambda} \lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda} \lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}. \end{aligned}$$

To obtain the MLE for λ, we first find the log-likelihood function

$$\ln(x_1, \ldots, x_n; \lambda) = -n\lambda + \ln(\lambda) \sum_{i=1}^{n} x_i - \ln(\prod_{i=1}^{n} x_i!).$$

Now we take the derivative of this with respect to λ and set it to 0.

$$\begin{aligned} \frac{d}{d\lambda} \ln L(x_1, \ldots, x_n; \lambda) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda} &\overset{set}{=} 0 \\ \hat{\lambda} &= \frac{\sum_{i=1}^{n} x_i}{n} \\ &= \text{observed value of } \overline{X}. \end{aligned}$$

Thus, the MLE can be written as

$$\hat{\Lambda} = \overline{X}.$$

Then the best λ = (2+3+1+4+3)/5 = 13/5

Question 4: (30 points)

Let $X_1, \ldots, X_4$ be a random sample from a $N(0, \theta^2)$ distribution. Suppose we observed $(x_1, x_2, x_3, x_4) = (2.1, -1.3, 0.3, -0.5)$. Find the likelihood function. What is the best estimate for θ?

Solution:

The likelihood function is

$$L(x;\sigma^2) = f_X(x;\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x)^2}.$$

The log-likelihood function is

$$\ln L(x;\sigma^2) = -\ln(2\pi)^{\frac{1}{2}} - \ln\sigma - \frac{x^2}{2\sigma^2}.$$

To find the MLE for σ we differentiate $\ln L(x;\sigma^2)$ with respect to σ and set it equal to zero.

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\ln L &= -\frac{1}{\sigma} + \frac{x^2}{\sigma^3} \\
&= -\frac{1}{\sigma} + \frac{x^2}{\sigma^3} \overset{set}{=} 0 \to \hat{\sigma}X^2 = \hat{\sigma}^3 \to \hat{\sigma} = |X|. \\
\frac{\partial^2}{\partial\sigma^2}\ln L &= \frac{1}{\sigma^2} - \frac{3x^2}{\sigma^4} < 0 \text{ when } \hat{\sigma} = |x|.
\end{aligned}$$