Assignment 3 - Key

An urn consists of 20 red balls and 30 green balls. We choose 10 balls at random from the urn. The sampling is done **without** replacement (repetition not allowed).

(a) What is the probability that there will be exactly 4 red balls among the chosen balls?

(b) Given that there are at least 3 red balls among the chosen balls, what is the probability that there are exactly 4 red balls?

Solution:

a)

$$P(A) = \frac{\binom{20}{4}\binom{30}{6}}{\binom{50}{10}}$$

b)

$$\begin{aligned} P(B|A) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)}{P(B)} \end{aligned}$$

$$P(B) = \sum_{k=3}^{10} \frac{\binom{20}{k}\binom{30}{10-k}}{\binom{50}{10}}$$

Therefore:

$$P(B|A) = \frac{\binom{20}{4}\binom{30}{6}}{\sum_{k=3}^{10}\binom{20}{k}\binom{30}{10-k}}$$

Suppose that $Y = -2X + 3$. If we know $EY = 1$ and $EY^2 = 9$, find EX and $\text{Var}(X)$.

Solution:

$$Y = -2X + 3$$

$$EY = -2EX + 3 \quad \text{linearity of expectation}$$

$$1 = -2EX + 3 \quad \rightarrow \qquad EX = 1$$

$$\text{Var}(Y) = 4 \times \text{Var}(X) = EY^2 - (EY)^2 = 9 - 1 = 8$$
$$\rightarrow \quad \text{Var}(X) = 2$$

The **median** of a random variable X is defined as any number m that satisfies both of the following conditions:

$$P(X \geq m) \geq \frac{1}{2} \qquad \text{and} \qquad P(X \leq m) \geq \frac{1}{2}.$$

Note that the median of X is not necessarily unique. Find the median of X if

(a) The PMF of X is given by

$$P_X(k) = \begin{cases} 0.4 & \text{for } k = 1 \\ 0.3 & \text{for } k = 2 \\ 0.3 & \text{for } k = 3 \\ 0 & \text{otherwise} \end{cases}$$

(b) X is the result of a rolling of a fair die.

(c) $X \sim Geometric(p)$, where $0 < p < 1$.

Solution: (a) $m = 2$, since

$$P(X \geq 2) = 0.6 \text{ and } P(X \leq 2) = 0.7$$

(b)

$$P_X(k) = \frac{1}{6} \text{ for } k = 1, 2, 3, 4, 5, 6$$
$$\rightarrow 3 \leq m \leq 4$$

Thus, we conclude $3 \leq m \leq 4$. Any value $\in [3, 4]$ is a median for X.

Part (c) has been dropped from the assignment. (assigned as bonus points)

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} x^2 + \frac{2}{3} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $E(X^n)$, for $n = 1, 2, 3, \cdots$.

(b) Find variance of X.

Solution:

(a)

Using LOTUS we have

$$\begin{aligned} E[X^n] &= \int_{-\infty}^{\infty} x^n f_X(x) dx \\ &= \int_0^1 x^n (x^2 + \frac{2}{3}) dx \\ &= \int_0^1 (x^{n+2} + \frac{2}{3} x^n) dx \\ &= \left[\frac{1}{n+3} x^{n+3} + \frac{2}{3(n+1)} x^{n+1} \right]_0^1 \\ &= \frac{1}{n+3} + \frac{2}{3(n+1)} \\ &= \frac{5n+9}{3(n+1)(n+3)}. \quad \text{where } n = 1, 2, 3, \cdots \end{aligned}$$

(b)

We know that

$$\text{Var}(X) = EX^2 - (EX)^2.$$

So we need to find the values of EX and EX^2

$$E[X] = \frac{7}{12}$$

$$E[X^2] = \frac{19}{45}$$

Thus, we have

$$\text{Var}(X) = EX^2 - (EX)^2 = \frac{19}{45} - (\frac{7}{12})^2 = 0.0819.$$

Let X be a continuous random variable with PDF

$$f_X(x) = \begin{cases} \frac{5}{32}x^4 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

and let $Y = X^2$.

(a) Find CDF of Y.

(b) Find PDF of Y.

(c) Find EY.

Solution:

(a) First, we note that $R_Y = [0, 4]$. As usual, we start with the CDF. For $y \in [0, 4]$, we have

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(X^2 \leq y) \\ &= P(0 \leq X \leq \sqrt{y}) \quad \text{since } x \text{ is not negative} \\ &= \int_0^{\sqrt{y}} \frac{5}{32} x^4 dx \\ &= \frac{1}{32}(\sqrt{y})^5 \\ &= \frac{1}{32} y^2 \sqrt{y} \end{aligned}$$

Thus, the CDF of Y is given by

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ \frac{1}{32} y^2 \sqrt{y} & \text{for } 0 \leq y \leq 4 \\ 1 & \text{for } y > 4. \end{cases}$$

(b)

$$f_Y(y) = F'_Y(y) = \begin{cases} \frac{5}{64} y \sqrt{y} & \text{for } 0 \leq y \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(c)

To find the EY, we can directly apply LOTUS,

$$\begin{aligned} E[Y] = E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x) dx \\ &= \int_0^2 x^2 \cdot \frac{5}{32} x^4 dx \\ &= \int_0^2 \frac{5}{32} x^6 dx \\ &= \frac{5}{32} \times \frac{1}{7} \times 2^7 = \frac{20}{7}. \end{aligned}$$

A company makes a certain device. We are interested in the lifetime of the device. It is estimated that around 2% of the devices are defected from the start so they have a lifetime of 0 years. If a device is not defected, then the lifetime of the device is exponentially distributed with parameter $\lambda = 2$ years. Let X be the lifetime of a randomly chosen device:

(a) Find the generalized PDF of X.

(b) Find $P(X \geq 1)$.

(c) Find $P(X > 2 | X \geq 1)$.

(d) Find EX and $\mathrm{Var}(X)$.

Solution:

(a)

$$f_X(x) = \frac{2}{100}\delta(x) + \frac{98}{100}f_Z(x) \quad \text{where } f_Z(x) \text{ is the} \sim Exponential(\lambda = 2) \text{ PDF}$$

Thus:

$$\begin{aligned} f_X(x) &= \frac{1}{50}\delta(x) + \frac{49}{50} \cdot 2e^{-2x}u(x) \\ &= \frac{1}{50}\delta(x) + \frac{49}{25} \cdot e^{-2x}u(x) \end{aligned}$$

(b)

$$P(X \geq 1) = \int_1^\infty f_X(x)dx = \frac{49}{50}\int_1^\infty f_Z(x)dx = \frac{49}{50}e^{-2}$$

(c)

$$\begin{aligned} P(X > 2 | X \geq 1) &= \frac{P(X > 2 \text{ and } X \geq 1)}{P(X \geq 1)} = \frac{P(X > 2)}{P(X \geq 1)} \\ &= \frac{\frac{49}{50}e^{-2\times 2}}{\frac{49}{50}e^{-2\times 1}} = e^{-2} \end{aligned}$$

(d)

$$\begin{aligned} EX &= \frac{1}{50} \cdot 0 + \frac{49}{50} \cdot EY \quad \text{ where } Y \sim Exponential(\lambda = 2) \\ &= \frac{49}{50} \cdot \frac{1}{2} = 0.49 \end{aligned}$$

$$\text{Var}(X) = EX^2 - (EX)^2 = EX^2 - (0.49)^2$$

$$
\begin{aligned}
EX^2 &= \frac{1}{50} \cdot 0 + \frac{49}{50} \cdot EY^2 \\
&= \frac{49}{50}(\frac{1}{\lambda^2} + \frac{1}{\lambda^2}) \quad \text{where } \lambda = 2 \\
&= \frac{49}{50}(\frac{1}{4} + \frac{1}{4}) = \frac{1}{2} \cdot \frac{49}{50}
\end{aligned}
$$

Thus:

$$
\mathrm{Var}(X) = (\frac{1}{2} \cdot \frac{49}{50}) - (0.49)^2 = 0.2499
$$