Assignment 2 – Key

Consider a random experiment with a sample space.

$$S = \{1, 2, 3, \cdots\}.$$

Suppose that we know:

$$P(k) = P(\{k\}) = \frac{c}{3^k} \quad \text{for} \quad k = 1, 2, \cdots$$

where c is a constant number.
(a) Find c.
(b) Find $P(\{2, 4, 6\})$.
(c) Find $P(\{3, 4, 5, \cdots\})$.

Solution:

(a) We must have

$$\sum_{k=1}^{\infty} P(k) = 1.$$

Thus

$$\begin{aligned} 1 &= \sum_{k=1}^{\infty} \frac{c}{3^k} \\ &= c \cdot \sum_{k=1}^{\infty} \frac{1}{3^k} \\ &= c \cdot \frac{\frac{1}{3}}{1 - \frac{1}{3}} \\ &= \frac{c}{2} \end{aligned}$$

Thus

$$c = 2$$

(b)

$$\begin{aligned} P(\{2,4,6\}) &= P(2)+P(4)+P(6) \\ &= 2(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}) \\ &= 2\cdot\frac{3^4+3^2+1}{3^6} \\ &= 2\cdot\frac{91}{3^6} \approx 0.25 \end{aligned}$$

(c)

$$\begin{aligned} P(\{3,4,5,\cdots\}) &= \sum_{k=3}^{\infty} P(k) = 2\sum_{k=3}^{\infty}\frac{1}{3^k} \\ &= 2\cdot\frac{\frac{1}{3^3}}{1-\frac{1}{3}} \\ &= \frac{1}{9} \end{aligned}$$

Let T be the time needed to complete a job at a certain factory. By using the historical data, we know that

$$P(T \le t) = \begin{cases} \frac{1}{16}t^2 & \text{for} \quad 0 \le t \le 4 \\ 1 & \text{for} \quad t \ge 4 \end{cases}$$

(a) Find the probability that the job is completed in less than one hour, i.e., find $P(T \le 1)$.
(b) Find the probability that the job needs more than 2 hours.
(c) Find the probability that $1 \le T \le 3$.

Solution:

(a)

$$\begin{aligned} P(T \le 1) &= \frac{1}{16}\cdot 1 \\ &= \frac{1}{16} \end{aligned}$$

(b)

$$\begin{aligned} P(T \geq 2) &= 1 - P(T < 2) \\ &= 1 - \frac{1}{16} \cdot 4 \\ &= 1 - \frac{1}{4} \\ &= \frac{3}{4} \end{aligned}$$

(c)

$$\begin{aligned} P(1 \leq T \leq 3) &= P(T \leq 3) - P(T \leq 1) \\ &= \frac{9}{16} - \frac{1}{16} \\ &= \frac{1}{2} \end{aligned}$$

Suppose that of all the customers at a coffee shop:

-70% purchase a cup of coffee.
-40% purchase a piece of cake.
-20% purchase both a cup of coffee and a piece of cake.
Given that a randomly chosen customer has purchased a piece of cake, what is the probability that he/she has also purchased a cup of coffee?

Solution:

We know

$$\begin{aligned} P(A) &= 0.7 \\ P(B) &= 0.4 \\ P(A \cap B) &= 0.2 \end{aligned}$$

Therefore:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{0.2}{0.4} \\ &= \frac{1}{2} \end{aligned}$$

A **real** number X is selected uniformly at random in the continuous interval $[0, 10]$. (For example, X could be 3.87.)
(a) Find $P(2 \leq X \leq 5)$.
(b) Find $P(X \leq 2 | X \leq 5)$.
(c) Find $P(3 \leq X \leq 8 | X \geq 4)$.

Solution:

Since X is selected uniformly at random we conclude that:

$$P(a \leq X \leq b) = (b - a) \times c \quad \text{for} \quad 0 \leq a \leq b \leq 10$$

where c is a constant. Since $P(0 \leq X \leq 10) = 1$. We conclude $c = \frac{1}{10}$. Therefore

$$P(a \leq X \leq b) = \frac{b-a}{10} \quad \text{for} \quad 0 \leq a \leq b \leq 10$$

(a)

$$\begin{aligned} P(2 \leq X \leq 5) &= \frac{5-2}{10} \\ &= 0.3 \end{aligned}$$

(b)

$$\begin{aligned} P[(X \leq 2)|(X \leq 5)] &= \frac{P[(X \leq 2) \cap (X \leq 5)]}{P[X \leq 5]} \\ &= \frac{P[(X \leq 2) \quad \text{and} \quad (X \leq 5)]}{P[X \leq 5]} \\ &= \frac{P[X \leq 2]}{[X \leq 5]} \\ &= \frac{P[0 \leq X \leq 2]}{P[0 \leq X \leq 5]} \\ &= \frac{\frac{2-0}{10}}{\frac{5-0}{10}} \\ &= 0.4 \end{aligned}$$

(c)

$$\begin{aligned} P[3 \le X \le 8 | X \ge 4] &= \frac{P[(3 \le X \le 8) \quad \text{and} \quad (X \ge 4)]}{P[X \ge 4]} \\ &= \frac{P[4 \le X \le 8]}{P[4 \le X \le 10]} \\ &= \frac{\frac{8-4}{10}}{\frac{10-4}{10}} \\ &= \frac{4}{6} = \frac{2}{3}. \end{aligned}$$

Let X and Y be two independent discrete random variables with the following PMFs:

$$P_X(k) = \begin{cases} \frac{1}{4} & \text{for } k = 1 \\ \frac{1}{8} & \text{for } k = 2 \\ \frac{1}{8} & \text{for } k = 3 \\ \frac{1}{2} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P_Y(k) = \begin{cases} \frac{1}{6} & \text{for } k = 1 \\ \frac{1}{6} & \text{for } k = 2 \\ \frac{1}{3} & \text{for } k = 3 \\ \frac{1}{3} & \text{for } k = 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find $P(X \le 2 \text{ and } Y \le 2)$.

(b) Find $P(X > 2 \text{ or } Y > 2)$.

(c) Find $P(X > 2 | Y > 2)$.

(d) Find $P(X < Y)$.

Solution:

(a) X and Y are two independent random variables. So:

$$\begin{aligned}P(X \leq 2 \text{ and } Y \leq 2) &= P(X \leq 2) \cdot P(Y \leq 2)\\ &= \big(P_X(1) + P_X(2)\big) \cdot \big(P_Y(1) + P_Y(2)\big)\\ &= (\frac{1}{4} + \frac{1}{8})(\frac{1}{6} + \frac{1}{6})\\ &= \frac{3}{8} \cdot \frac{1}{3} = \frac{1}{8}\end{aligned}$$

(b) Using $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and the fact that X and Y are two independent random variables:

$$\begin{aligned}P(X > 2 \text{ or } Y > 2) &= P(X > 2) + P(Y > 2) - P(X > 2 \text{ and } Y > 2)\\ &= (\frac{1}{8} + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{3}) - (\frac{1}{8} + \frac{1}{2}) \cdot (\frac{1}{3} + \frac{1}{3})\\ &= \frac{5}{8} + \frac{2}{3} - \frac{5}{8} \cdot \frac{2}{3} = \frac{21}{24} = \frac{7}{8}\end{aligned}$$

(c) Using the law of total probability:

$$\begin{aligned}P(X < Y) &= \sum_{k=1}^{4} P(X < Y | Y = k) \cdot P(Y = k)\\ &= P(X < 1 | Y = 1) \cdot P(Y = 1) + P(X < 2 | Y = 2) \cdot P(Y = 2)\\ &+ P(X < 3 | Y = 3) \cdot P(Y = 3) + P(X < 4 | Y = 4) \cdot P(Y = 4)\\ &= 0 \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{6} + (\frac{1}{4} + \frac{1}{8}) \cdot \frac{1}{3} + (\frac{1}{4} + \frac{1}{8} + \frac{1}{8}) \cdot \frac{1}{3}\\ &= \frac{1}{24} + \frac{1}{8} + \frac{1}{6} = \frac{8}{24} = \frac{1}{3}\end{aligned}$$

Question 7: Geometric Distribution (15 points)

For X ~ Geo (0.20), find P (X > 5), P (2 < X ≤ 6) and P (X > 5 | X < 8)

Solution:

First note that if $R_X \subset \{0,1,2,\cdots\}$, then

- $P(X>5) = \sum_{k=6}^{\infty} P_X(k) = 1 - \sum_{k=0}^{5} P_X(k)$.
- $P(2<X\leq 6) = P_X(3)+P_X(4)+P_X(5)+P_X(6)$.
- $P(X>5|X<8) = \frac{P(5<X<8)}{P(X<8)} = \frac{P_X(6)+P_X(7)}{\sum_{k=0}^{7} P_X(k)}$.

So,

(a) $X \sim Geometric(\frac{1}{5}) \longrightarrow \quad P_X(k) = (\frac{4}{5})^{k-1}(\frac{1}{5}) \quad$ for $k=1,2,3,\cdots$

Therefore,

$$\begin{aligned}
P(X>5) &= 1 - \sum_{k=1}^{5} (\frac{4}{5})^{k-1}(\frac{1}{5}) \\
&= 1 - (\frac{1}{5}) \cdot (1 + (\frac{4}{5}) + (\frac{4}{5})^2 + (\frac{4}{5})^3 + (\frac{4}{5})^4) \\
&= 1 - (\frac{1}{5}) \cdot \frac{1-(\frac{4}{5})^5}{1-(\frac{4}{5})} = (\frac{4}{5})^5.
\end{aligned}$$

Note that we can obtain this result directly from the random experiment behind the geometric random variable:

$P(X<5) = P(\text{No heads in 5 coin tosses}) = (\frac{4}{5})^5$

$$\begin{aligned}
P(2<X\leq 6) &= P_X(3)+P_X(4)+P_X(5)+P_X(6) \\
&= (\frac{1}{5})(\frac{4}{5})^2 + (\frac{1}{5})(\frac{4}{5})^3 + (\frac{1}{5})(\frac{4}{5})^4 + (\frac{1}{5})(\frac{4}{5})^5 \\
&= (\frac{1}{5})(\frac{4}{5})^2 \cdot (1 + \frac{4}{5} + (\frac{4}{5})^2 + (\frac{4}{5})^3) \\
&= (\frac{4}{5})^2(1-(\frac{4}{5})^4).
\end{aligned}$$

$$P(X > 5 | X < 8) = \frac{P(5 < X < 8)}{P(X < 8)} = \frac{P_X(6) + P_X(7)}{\sum_{k=1}^{7} P_X(k)}$$
$$= \frac{(\frac{1}{5})((\frac{4}{5})^5 + (\frac{4}{5})^6)}{(\frac{1}{5}) \sum_{k=1}^{7} (\frac{4}{5})^{k-1}}$$
$$= \frac{(\frac{4}{5})^5 + (\frac{4}{5})^6}{1 + (\frac{4}{5}) + \cdots (\frac{4}{5})^6}$$

Question 8: Normal Distribution (10 points)

For X ~ N (4.3, 1.96), Use Normal Tables to find P (X ≤ 5), P (2.8 ≤ X ≤ 5.5) and P (X ≥ 4.3)

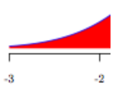
Solution

$\mu = 4.3$, $\sigma^2 = 1.96$, and $\sigma = (1.96)^{0.5} = 1.4$

z = (x – μ) / σ = (x – 4.3) / 1.4

For P (X ≤ 5), we find z = (5 – 4.3) / 1.4 = 0.70 / 1.4 = 1 / 2 = 0.50

From the table P (X ≤ 5) = 0.6915



	0.00	0.01	0.0
0.0	0.5000	0.5040	0.508
0.1	0.5398	0.5438	0.547
0.2	0.5793	0.5832	0.587
0.3	0.6179	0.6217	0.625
0.4	0.6554	0.6591	0.662
0.5	0.6915	0.6950	0.698
0.6	0.7257	0.7291	0.732
0.7	0.7580	0.7611	0.764
0.8	0.7881	0.7910	0.793
0.9	0.8159	0.8186	0.821

P (2.8 ≤ X ≤ 5.5) = P (X ≤ 5.5) - P (X ≤ 2.8)

For P (X ≤ 5.5), we find z = (5.5 – 4.3) / 1.4 = 1.20 / 1.4 = 0.857

From the Standard Normal Table, we have values for 0.850 (= 0.8023) and 0.860 (= 0.8051). To find the value for 0.857 we need to interpolate.

0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.71
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.75
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.78
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.81
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.83
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.85
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.88

51 – 23 = 28, so 7(28/10) = 19.6. Thus 0.857 (=0.8023 + 0.0019 = 0.8042)

Resulting in P (X ≤ 5.5) = 0.8042

For P (X ≤ 2.8), we find z = (2.8 – 4.3) / 1.4 = - 1.50 / 1.4 = - 1.071

We can either use a table that has negative values or use symmetry properties and the current table

-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148

We have values for -1.07 (= 0.1423) and -1.08 (= 0.1401). However, -1.071 is extremely close to -1.071 and we can use 0.1423, but because this case involves negative values let us do it for practice.

01 – 23 = - 22, so 1(- 22/10) = -2.2. Thus – 1.701 (0.1423 – 0.0022 = 0.14208~ 0.1421)

P (2.8 ≤ X ≤ 5.5) = P (X ≤ 5.5) - P (X ≤ 2.8) = 0.857 – 0.1421 = 0.7149

What about P (X ≥ 4.3)? Note that 4.3 is the same mean value and because of the symmetry we have P (X ≥ 4.3) = P (X ≤ 4.3) = 0.50